Mikhail Itskov

Tensor Algebra and Tensor Analysis for Engineers

With Applications to Continuum Mechanics

Second Edition

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Preface to the Second Edition

This second edition is completed by a number of additional examples and exercises. In response of comments and questions of students using this book, solutions of many exercises have been improved for a better understanding. Some changes and enhancements are concerned with the treatment of skew-symmetric and rotation tensors in the first chapter. Besides, the text and formulae have thoroughly been reexamined and improved where necessary.

Aachen, January 2009 *Mikhail Itskov*

Preface to the First Edition

Like many other textbooks the present one is based on a lecture course given by the author for master students of the RWTH Aachen University. In spite of a somewhat difficult matter those students were able to endure and, as far as I know, are still fine. I wish the same for the reader of the book.

Although the present book can be referred to as a textbook one finds only little plain text inside. I tried to explain the matter in a brief way, nevertheless going into detail where necessary. I also avoided tedious introductions and lengthy remarks about the significance of one topic or another. A reader interested in tensor algebra and tensor analysis but preferring, however, words instead of equations can close this book immediately after having read the preface.

The reader is assumed to be familiar with the basics of matrix algebra and continuum mechanics and is encouraged to solve at least some of numerous exercises accompanying every chapter. Having read many other texts on mathematics and mechanics I was always upset vainly looking for solutions to the exercises which seemed to be most interesting for me. For this reason, all the exercises here are supplied with solutions amounting a substantial part of the book. Without doubt, this part facilitates a deeper understanding of the subject.

As a research work this book is open for discussion which will certainly contribute to improving the text for further editions. In this sense, I am very grateful for comments, suggestions and constructive criticism from the reader. I already expect such criticism for example with respect to the list of references which might be far from being complete. Indeed, throughout the book I only quote the sources indispensable to follow the exposition and notation. For this reason, I apologize to colleagues whose valuable contributions to the matter are not cited.

Finally, a word of acknowledgment is appropriate. I would like to thank Uwe Navrath for having prepared most of the figures for the book. Further, I am grateful to Alexander Ehret who taught me first steps as well as some “dirty” tricks in LaTeX, which were absolutely necessary to bring the manuscript to a printable form. He and Tran Dinh Tuyen are also acknowledged for careful proof reading and critical comments to an earlier version of the book. My special thanks go to the Springer-Verlag and in particular to Eva Hestermann-Beyerle and Monika Lempe for their friendly support in getting this book published.

Aachen, November 2006 *Mikhail Itskov*

Contents

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Vectors and Tensors in a Finite-Dimensional Space

1.1 Notion of the Vector Space

We start with the definition of the vector space over the field of real numbers $\mathbb{R}$.

Definition 1.1. *A vector space is a set $\mathbb{V}$ of elements called vectors satisfying the following axioms.*

A. To every pair, $\boldsymbol{x}$ and $\boldsymbol{y}$ of vectors in $\mathbb{V}$ there corresponds a vector $\boldsymbol{x}+\boldsymbol{y}$, called the sum of $\boldsymbol{x}$ and $\boldsymbol{y}$, such that

(A.1) $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ (addition is commutative),

(A.2) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ (addition is associative),

(A.3) there exists in $\mathbb{V}$ a unique vector zero $\boldsymbol{0}$, such that $\boldsymbol{0}+\boldsymbol{x}=\boldsymbol{x}$, $\forall \boldsymbol{x}\in\mathbb{V}$,

(A.4) to every vector $\boldsymbol{x}$ in $\mathbb{V}$ there corresponds a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x}+(-\boldsymbol{x})=\boldsymbol{0}$.

B. To every pair α and $\boldsymbol{x}$, where α is a scalar real number and $\boldsymbol{x}$ is a vector in $\mathbb{V}$, there corresponds a vector $\alpha\boldsymbol{x}$, called the product of α and $\boldsymbol{x}$, such that

(B.1) $\alpha(\beta\boldsymbol{x})=(\alpha\beta)\boldsymbol{x}$ (multiplication by scalars is associative),

(B.2) $1\boldsymbol{x}=\boldsymbol{x}$,

(B.3) $\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ (multiplication by scalars is distributive with respect to vector addition),

(B.4) $(\alpha+\beta)\boldsymbol{x}=\alpha\boldsymbol{x}+\beta\boldsymbol{x}$ (multiplication by scalars is distributive with respect to scalar addition),

$\forall\alpha,\beta\in\mathbb{R}$, $\forall\boldsymbol{x},\boldsymbol{y}\in\mathbb{V}$.

Examples of vector spaces.

1) The set of all real numbers $\mathbb{R}$.

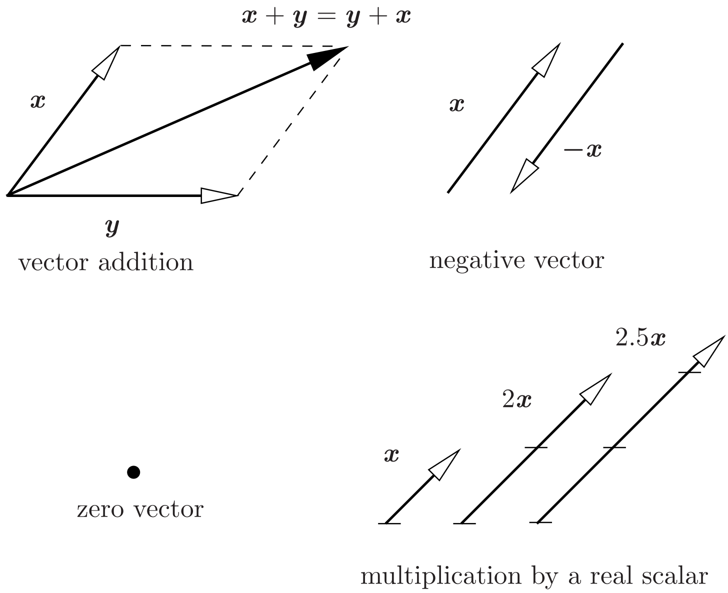

Fig. 1.1. Geometric illustration of vector axioms in two dimensions

2) The set of all directional arrows in two or three dimensions. Applying the usual definitions for summation, multiplication by a scalar, the negative and zero vector (Fig. 1.1) one can easily see that the above axioms hold for directional arrows.
3) The set of all n-tuples of real numbers $\mathbb{R}$:

$$\boldsymbol{a} = \begin{Bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{Bmatrix}.$$

Indeed, the axioms (A) and (B) apply to the n-tuples if one defines addition, multiplication by a scalar and finally the zero tuple, respectively, by

$$\boldsymbol{a}+\boldsymbol{b} = \begin{Bmatrix} a_1+b_1 \\ a_2+b_2 \\ . \\ . \\ a_n+b_n \end{Bmatrix}, \quad \alpha\boldsymbol{a} = \begin{Bmatrix} \alpha a_1 \\ \alpha a_2 \\ . \\ . \\ \alpha a_n \end{Bmatrix}, \quad \boldsymbol{0} = \begin{Bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{Bmatrix}.$$

4) The set of all real-valued functions defined on a real line.

1.2 Basis and Dimension of the Vector Space

Definition 1.2. *A set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is called linearly dependent if there exists a set of corresponding scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$, not all zero, such that*

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0}. \tag{1.1}$$

Otherwise, the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are called linearly independent. In this case, none of the vectors $\boldsymbol{x}_i$ is the zero vector (Exercise 1.2).

Definition 1.3. *The vector*

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i \tag{1.2}$$

is called linear combination of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, where $\alpha_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$).

Theorem 1.1. *The set of n non-zero vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ is linearly dependent if and only if some vector $\boldsymbol{x}_k$ $(2 \le k \le n)$ is a linear combination of the preceding ones $\boldsymbol{x}_i$ $(i = 1, \ldots, k-1)$.*

Proof. If the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are linearly dependent, then

$$\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i = \boldsymbol{0},$$

where not all α_i are zero. Let α_k $(2 \le k \le n)$ be the last non-zero number, so that $\alpha_i = 0$ $(i = k+1, \ldots, n)$. Then,

$$\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{x}_k = \sum_{i=1}^{k-1} \frac{-\alpha_i}{\alpha_k} \boldsymbol{x}_i.$$

Thereby, the case $k = 1$ is avoided because $\alpha_1 \boldsymbol{x}_1 = \boldsymbol{0}$ implies that $\boldsymbol{x}_1 = \boldsymbol{0}$ (Exercise 1.1). Thus, the sufficiency is proved. The necessity is evident.

Definition 1.4. *A basis of a vector space $\mathbb{V}$ is a set $\mathcal{G}$ of linearly independent vectors such that every vector in $\mathbb{V}$ is a linear combination of elements of $\mathcal{G}$. A vector space $\mathbb{V}$ is finite-dimensional if it has a finite basis.*

Within this book, we restrict our attention to finite-dimensional vector spaces. Although one can find for a finite-dimensional vector space an infinite number of bases, they all have the same number of vectors.

Theorem 1.2. *All the bases of a finite-dimensional vector space $\mathbb{V}$ contain the same number of vectors.*

Proof. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ and $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m\}$ be two arbitrary bases of $\mathbb{V}$ with different numbers of elements, say $m > n$. Then, every vector in $\mathbb{V}$ is a linear combination of the following vectors:

$$\boldsymbol{f}_1, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n. \tag{1.3}$$

These vectors are non-zero and linearly dependent. Thus, according to Theorem 1.1 we can find such a vector $\boldsymbol{g}_k$, which is a linear combination of the preceding ones. Excluding this vector we obtain the set $\mathcal{G}'$ by

$$\boldsymbol{f}_1, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}_{k+1}, \ldots, \boldsymbol{g}_n$$

again with the property that every vector in $\mathbb{V}$ is a linear combination of the elements of $\mathcal{G}'$. Now, we consider the following vectors

$$\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{k-1}, \boldsymbol{g}_{k+1}, \ldots, \boldsymbol{g}_n$$

and repeat the excluding procedure just as before. We see that none of the vectors $\boldsymbol{f}_i$ can be eliminated in this way because they are linearly independent. As soon as all $\boldsymbol{g}_i$ $(i = 1, 2, \ldots, n)$ are exhausted we conclude that the vectors

$$\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_{n+1}$$

are linearly dependent. This contradicts, however, the previous assumption that they belong to the basis $\mathcal{F}$.

Definition 1.5. *The dimension of a finite-dimensional vector space $\mathbb{V}$ is the number of elements in a basis of $\mathbb{V}$.*

Theorem 1.3. *Every set $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n\}$ of linearly independent vectors in an n-dimensional vectors space $\mathbb{V}$ forms a basis of $\mathbb{V}$. Every set of more than n vectors is linearly dependent.*

Proof. The proof of this theorem is similar to the preceding one. Let $\mathcal{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ be a basis of $\mathbb{V}$. Then, the vectors (1.3) are linearly dependent and non-zero. Excluding a vector $\boldsymbol{g}_k$ we obtain a set of vectors, say $\mathcal{G}'$, with the property that every vector in $\mathbb{V}$ is a linear combination of the elements of $\mathcal{G}'$. Repeating this procedure we finally end up with the set $\mathcal{F}$ with the same property. Since the vectors $\boldsymbol{f}_i$ $(i = 1, 2, \ldots, n)$ are linearly independent they form a basis of $\mathbb{V}$. Any further vectors in $\mathbb{V}$, say $\boldsymbol{f}_{n+1}, \boldsymbol{f}_{n+2}, \ldots$ are thus linear combinations of $\mathcal{F}$. Hence, any set of more than n vectors is linearly dependent.

Theorem 1.4. *Every set $\mathcal{F} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_m\}$ of linearly independent vectors in an n-dimensional vector space $\mathbb{V}$ can be extended to a basis.*