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Preface

A functional differential equation (FDE) describes the evolution of a dynamical system for which the rate of change of the state variable depends on not only the current but also the historical and even future states of the system. FDEs arise naturally in economics, life sciences, and engineering, and the study of FDEs has been a major source of inspiration for advancement of nonlinear analysis and infinite-dimensional dynamical systems. Therefore, FDEs provide an excellent theoretical platform for developing an interdisciplinary approach to understanding complex nonlinear phenomena via appropriate mathematical techniques.

Unfortunately, the study of FDEs is difficult for newcomers, since a background in nonlinear analysis, ordinary differential equations, and dynamical systems is a prerequisite. On the other hand, the novelty and challenge of fundamental research in the field of FDEs has often been underappreciated. This is especially so in our effort to describe the qualitative behaviors of solutions near equilibria or periodic orbits: these qualitative behaviors can be derived from those of finite-dimensional (ordinary differential) systems obtained through a center and center-unstable manifold reduction process, and hence the (local) bifurcation theory that deals with significant changes in these qualitative behaviors is in principle a consequence of the corresponding theory for finite-dimensional (ordinary differential) systems. The highly nontrivial and often lengthy calculation of center manifold reduction, however, not only leads to enormous duplication of calculation efforts, but also prevents us from discovering simple and key mechanisms behind observed bifurcation phenomena due to the infinite-dimensionality of FDEs. This, in turn, makes it difficult to express bifurcation results explicitly in terms of model parameters and to compare and validate different results. Another challenge is the study of the birth and global continuation of bifurcation of periodic solutions and the coexistence of multiple periodic solutions when the parameters are far from the bifurcation/critical values. There has been substantial progress dedicated to this global bifurcation problem, and remarkably, the presence of a delayed or advanced argument in the nonlinearity can sometimes facilitate the application of topological methods such as equivalent degrees to examine the global continua of branches of periodic solutions, and this has inspired interesting developments in the spectral analysis of circulant matrices.

On the other hand, the study of dynamical systems with symmetries has become well established as a major branch of nonlinear systems theory. The current interest in the field dates mainly to the appearance of the equivariant branching lemma of Vanderbauwhede and Cicogna and the equivariant Hopf bifurcation theorem of Golubitsky and Stewart, both of which are reviewed in the book by Golubitsky, Stewart and Schaeffer. Since then, important new theories have been developed for more complex dynamical phenomena, including the existence, stability, and bifurcations of systems of heteroclinic connections, and the symmetry groups and bifurcations of chaotic attractors.

To a large extent, the phenomenal growth in the subject has been due to its effectiveness in explaining the bifurcations and dynamical phenomena that are seen in a wide range of physical systems including coupled oscillators, reaction–diffusion systems, convecting fluids, and mechanical systems. A local symmetric bifurcation theory for FDEs can be derived from that of but since some special properties of spatiotemporal symmetry of FDEs may be reflected generically in the reduced finite-dimensional systems, one can and should make general observations about the particular bifurcation patterns of symmetric FDEs.

The purpose of this book is to summarize some practical and general approaches and frameworks for the investigation of bifurcation phenomena of FDEs depending on parameters. The book aims to be self-contained, so the reader should find in this book all relevant materials on bifurcation, dynamical systems with symmetry, functional differential equations, normal forms, and center manifold reduction. This material was used in graduate courses on functional differential equations at Hunan University (China) and York University (Canada). We want to thank all students in these courses for their careful reading and some helpful comments. We would like especially to thank Dr. Jing Fang and Dr. Xiang-Sheng Wang for their careful reading of an early version of the manuscript and for their critical comments.

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Introduction to Dynamic Bifurcation Theory

1.1 Introduction

The change in the qualitative behavior of solutions as a control parameter (or control parameters) in a system is varied and is known as a *bifurcation*. When the solutions are restricted to neighborhoods of a given equilibrium, a bifurcation occurs often when the zero solution of the linearization of the system at the equilibrium changes its stability. To illustrate the basic concepts of bifurcation phenomena, we consider the following continuous dynamical system defined by the C^r $(r \geq 1)$ vector field f: $\Lambda \times U \to \mathbb{R}^n$:

$$\dot{x} = f(\mu, x), \quad \mu \in \Lambda \subseteq \mathbb{R}^m, \quad x \in U \subseteq \mathbb{R}^n, \tag{1.1}$$

where U and Λ are open sets, x is the state variable, and μ is the (bifurcation) parameter.

Continuously varying μ may change the qualitative behavior of the solutions of (1.1). A value $\mu \in \Lambda$ for which such a change occurs is called a *bifurcation* (*critical*) *value*. The set of all bifurcation values is called the *bifurcation set* in the parameter space $\mathbb{R}^m$. We may use a bifurcation diagram to schematically show the considered solutions (equilibria/fixed points, closed orbits/periodic orbits, invariant tori) of a system as a function of a bifurcation parameter in the system. It is normal to represent stable solutions with solid lines and unstable solutions with dashed lines.

Local bifurcations are relevant to the birth or initiation of bifurcations when the bifurcation parameter is close to a bifurcation value. A local bifurcation from a given solution (an equilibrium, a periodic orbit, etc.) can normally be detected from a local stability analysis at the given solution. The global bifurcation thereby concerns the continuation of a local bifurcation when the bifurcation parameter is away from the bifurcation value.

The bifurcation phenomena is linked closely to the concepts of topological equivalence, structural stability, and genericity, which are described in the next section.

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1.2 Topological Equivalence

In the study of dynamical systems, we are interested in not only specific solutions of a specific system, but also classification of solutions of a particular system and classification of systems according to general qualitative behaviors, that is, the number, position, and stability of equilibria, periodic orbits, and other isolated invariant sets.

In what follows, we will not distinguish a flow and a dynamical system. This means that we consider a continuous mapping $\Phi\colon \mathbb{R}\times U \to U$ over an open set $U \subseteq \mathbb{R}^n$ such that $\Phi(0,x)=x$ and $\Phi(t,\Phi(s,x)) = \Phi(t+s,x)$ for t, $s\in\mathbb{R}$, and $x\in U$. Sometimes, we write it as $\Phi^t := \Phi(t,\cdot)\colon U\to U$ for $t\in\mathbb{R}$.

We consider two dynamical systems to be (locally) equivalent if their (local) phase portraits are similar in a qualitative sense, that is, if they can be locally transformed into each other through a continuous transformation. More precisely, we introduce the following definition.

Definition 1.1. A dynamical system Φ in $\mathbb{R}^n$ is said to be *topologically equivalent* in a region $U\subset\mathbb{R}^n$ to a dynamical system Ψ in a region $V\subset\mathbb{R}^n$ if there exists a homeomorphism $h\colon U\to V$ that maps the orbits of Φ in U onto the orbits of Ψ in V, preserving the direction of time.

A homeomorphism is an invertible map such that both the map and its inverse are continuous. A homomorphism is called a diffeomorphism if it is C^1-smooth and its inverse is also C^1-smooth. The definition of topological equivalence can be generalized to cover more general cases in which the state space is a complete metric or, in particular, a Banach space. The definition also remains meaningful when the state space is a smooth finite-dimensional manifold in $\mathbb{R}^n$, for example, a two-dimensional torus $\mathbb{T}^2$ or sphere $\mathbb{S}^2$. The phase portraits of topologically equivalent systems are often said to be topologically equivalent.

Example 1.1. Consider the flows Φ^t and Ψ^t associated with the differential equations

$$\dot{x} = -x \quad \text{and} \quad \dot{y} = -3y,$$

respectively. The homeomorphism $h\colon\mathbb{R}\to\mathbb{R}$ given by $h(x)=x^3$ for $x\in\mathbb{R}$ maps the orbits of Φ onto those of Ψ.

Definition 1.2. Two flows Φ^t (on U) and Ψ^t (on V) are called *topologically conjugate* if there exists a homeomorphism $h\colon U\to V$ such that

$$\Psi^t = h\circ\Phi^t\circ h^{-1} \quad \text{for} \quad t\in\mathbb{R}.$$

We also use the term *smoothly conjugate* (or *diffeomorphic*) if the involved homeomorphism is a diffeomorphism and the flows are smooth.

For example, for a continuous-time system

$$\dot{x} = f(x), \quad x\in\mathbb{R}^n, \tag{1.2}$$

if h is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, and $x = h(y)$, then the system

$$\dot{y} = g(y), \quad y \in \mathbb{R}^n \tag{1.3}$$

with $g(y) = [Dh(y)]^{-1} f(h(y))$ for all $y \in \mathbb{R}^n$ is smoothly equivalent (or diffeomorphic) to system (1.2). In fact, denoting by $\Phi^t(x)$ the flow associated with system (1.2), and letting $\Psi^t(y) = h^{-1}(\Phi^t(h(y)))$, we have

$$Dh(\Psi^t(y)) \frac{\mathrm{d}}{\mathrm{d}t} \Psi^t(y) = f(\Phi^t(h(y))),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi^t(y) = [Dh(\Psi^t(y))]^{-1} f(\Phi^t(h(y))) = g(\Psi^t(y)),$$

which implies that $\Psi^t(y)$ is the flow associated with system (1.3). Therefore, systems (1.2) and (1.3) are smoothly equivalent (or diffeomorphic).

In what follows, if the degree of smoothness of h is of interest, we also use the term C^k-equivalent or C^k-diffeomorphic.

Two diffeomorphic systems are practically identical and can be viewed as the same system written using different coordinates. Two diffeomorphic systems have similar qualitative behaviors. For such systems, the eigenvalues of corresponding equilibria are the same: Let x_0 and $y_0 = h(x_0)$ be such equilibria and let $A(x_0)$ and $B(y_0)$ denote corresponding Jacobian matrices. Then we have

$$A(x_0) = M^{-1}(x_0) B(y_0) M(x_0),$$

where $M(x) = Dh(x)$. Therefore, the characteristic polynomials for the matrices $A(x_0)$ and $B(y_0)$ coincide.

It is easy to construct nondiffeomorphic but topologically equivalent flows. To see this, consider a smooth scalar position function $\mu \colon \mathbb{R}^n \to (0, \infty)$ and assume that the right-hand sides of (1.2) and (1.3) are related by

$$f(x) = \mu(x) g(x) \quad \text{for} \quad x \in \mathbb{R}^n. \tag{1.4}$$

Then systems (1.2) and (1.3) are topologically equivalent since their orbits are identical, and it is the velocity of the motion that makes them different. Thus, the homeomorphism h in Definition 1.1 is the identity map $h(x) = x$. In other words, these two systems are distinguished only by the time parameterization along the orbits. We say that two systems (1.2) and (1.3) satisfying (1.4) for a smooth positive function μ are *orbitally equivalent*. Usually, two orbitally equivalent systems can be nondiffeomorphic, having cycles that look like the same closed curve in the phase space but different periods. For example, the system

$$\dot{r} = r(1 - r), \quad \dot{\theta} = 1$$

and the system

$$\dot{\rho} = 2\rho(1 - \rho), \quad \dot{\varphi} = 2$$

in $\mathbb{R}^2$ using polar coordinates are topologically equivalent, but not topologically conjugate, because their periodic orbits $r = 1$ and $\rho = 1$ have periods 2π and π, respectively.

Let x_0 be an equilibrium of the system (1.2), that is, $f(x_0) = 0$, and let A denote the Jacobian matrix $Df(x)$ evaluated at $x = x_0$. Let n_-, n_0, and n_+ be the numbers of eigenvalues of A (counting multiplicities) with negative, zero, and positive real part, respectively. Recall that an equilibrium is called *hyperbolic* if $n_0 = 0$, that is, if A has no purely imaginary eigenvalues. A hyperbolic equilibrium is called a *hyperbolic saddle* if $n_-n_+ \neq 0$.

Topological equivalence of linear systems is generally easy to determine. If the linearized flow near an equilibrium is asymptotically stable, then the equilibrium is asymptotically stable. Moreover, two asymptotically stable n-dimensional linear flows are topologically equivalent.

Example 1.2. Consider two linear planar systems:

$$\dot{x} = -x, \quad \dot{y} = -y, \tag{1.5}$$

and

$$\dot{x} = -x - y, \quad \dot{y} = x - y. \tag{1.6}$$

Clearly, the origin is a stable equilibrium in both systems. All other trajectories of (1.5) are straight lines, while those of (1.6) are spirals. The equilibrium of the first system is a node, while in the second systems it is a focus. These two systems are neither orbitally nor smoothly equivalent. However, they are topologically equivalent.

We can further claim that *near a hyperbolic equilibrium p, the system behaves essentially like the linearized one*. In other words, Φ^t is topologically equivalent to $e^{Df(p)t}$ in a sufficiently small neighborhood of a hyperbolic equilibrium p (Grobman–Hartman theorem). See Grobman [123], Hartman [161, 162], Hirsch [163], Hale and Kocak [152] for details. As a result, determining topological equivalence near hyperbolic equilibria boils down to counting the dimensions of the local stable and unstable subspaces (manifolds).

Theorem 1.1. *Two systems of differential equations with hyperbolic equilibria are topologically equivalent near these equilibria if and only if their linearizations have the same number n_+ of eigenvalues with positive real parts and the same number n_- of eigenvalues with negative real parts.*

1.3 Structural Stability

There are dynamical systems whose phase portrait (in some domain) does not change qualitatively under all sufficiently small perturbations. For example, suppose that (1.1) has an equilibrium x_0 when $\mu = \mu_0$, that is,

$$f(\mu_0, x_0) = 0. \tag{1.7}$$